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INVARIANT NORMALIZATION OF NON-AUTONOMOUS HAMILTONIAN SYSTEMS†

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A new method of constructing canonical replacements of variables in parametric form, which differs from the existing constructive methods in the Hamiltonian procedure: the method of derivative functions and the method of generators, is proposed. A criterion of the existence of a parametric representation of the canonical replacement of variables is formulated and the law of the conversion of the Hamiltonian is derived. The method is used to obtain the normal form of Hamiltonians. A definition of the normal form [1, 2] is used which does not require separation into autonomous – non-autonomous and resonance – non-resonance cases and is carried out within a single approach. A system of equations, similar to the chain of equations obtained previously in [1, 2], is derived for the asymptotics of the normal form. Instead of the generator and generating Hamiltonian method a parameterized generating function is used [3], which enables, as in [1, 2], a chain of equations to be obtained directly for the non-autonomous Hamiltonians but without reducing the system to an autonomous form. © 2004 Elsevier Ltd. All rights reserved.

1. THE PARAMETRIC FORM OF THE CANONICAL TRANSFORMATIONS

We will formulate the general result of the parametrization of the canonical replacement of variables in Hamiltonian systems in the form of a theorem [3].

Theorem 1. Suppose the transformation of variables $\mathbf{q}, \mathbf{p} \to \mathbf{Q}, \mathbf{P}$ is written in the parametric form

$$\begin{aligned} \mathbf{q} &= \mathbf{x} - \frac{1}{2}\Psi_y, \quad \mathbf{Q} = \mathbf{x} = \frac{1}{2}\Psi_y \\ \mathbf{p} &= \mathbf{y} + \frac{1}{2}\Psi_x, \quad \mathbf{P} = \mathbf{y} - \frac{1}{2}\Psi_x \end{aligned} \tag{1.1}$$

Then, for any function $\Psi(t, \mathbf{x}, \mathbf{y})$

(1) the Jacobian of the two transformations $\mathbf{q} = \mathbf{q}(t, \mathbf{x}, \mathbf{y})$, $\mathbf{p} = \mathbf{p}(t, \mathbf{x}, \mathbf{y})$ and $\mathbf{Q} = \mathbf{Q}(t, \mathbf{x}, \mathbf{y})$, $\mathbf{P} = \mathbf{P}(t, \mathbf{x}, \mathbf{y})$ are identical:

$$\frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\mathbf{x}, \mathbf{y})} = \frac{\partial(\mathbf{Q}, \mathbf{P})}{\partial(\mathbf{x}, \mathbf{y})} = J(t, \mathbf{x}, \mathbf{y}) \tag{1.2}$$

(2) in the region $J > 0$ the transformation (1.1) of the variables $\mathbf{q}, \mathbf{p} \to \mathbf{Q}, \mathbf{P}$ converts the Hamiltonian system $H = H(t, \mathbf{q}, \mathbf{p})$ into the Hamiltonian system $\tilde{H} = \tilde{H}(t, \mathbf{Q}, \mathbf{P})$ as follows:

$$\Psi_t(t, \mathbf{x}, \mathbf{y}) + H(t, \mathbf{q}, \mathbf{p}) = \tilde{H}(t, \mathbf{Q}, \mathbf{P}) \tag{1.3}$$

where the arguments $\mathbf{q}, \mathbf{p}$, and $\mathbf{Q}, \mathbf{P}$ in Hamiltonians H and $\tilde{H}$ are expressed in terms of the parameters $\mathbf{x}$ and $\mathbf{y}$ by formulae (1.1).

Our aim is to investigate for what canonical transformations the parameterization exists.

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2. THE GENERATING FUNCTIONS

The canonical transformation can also be represented in terms of the generating functions $S_1(t, \mathbf{q}, \mathbf{P})$ and $S_2(t, \mathbf{Q}, \mathbf{p})$

$$dS_1 = \mathbf{p}d\mathbf{q} + \mathbf{Q}d\mathbf{P} + (\tilde{H} - H)dt, \quad \det S_{1\mathbf{qP}} \neq 0$$

$$dS_2 = -\mathbf{q}d\mathbf{p} + \mathbf{P}d\mathbf{Q} + (\tilde{H} - H)dt, \quad \det S_{2\mathbf{pQ}} \neq 0$$

We will introduce the new generating function

$$\Phi = \frac{1}{2}[S_1(t, \mathbf{q}, \mathbf{P}) - \mathbf{qP} + S_2(t, \mathbf{Q}, \mathbf{p}) + \mathbf{Qp}] \tag{2.1}$$

Its differential form is

$$d\Phi = \frac{1}{2}\sum_{i=1}^{n} \begin{vmatrix} Q_i - q_i & P_i - p_i \\ dQ_i + dq_i & dP_i + dp_i \end{vmatrix} + (\tilde{H} - H)dt \tag{2.2}$$

when $dt = 0$ the differential form $d\Phi$ was derived by Poincaré (see [5, p. 337]) and showed that if $\mathbf{Q}(\mathbf{q}, \mathbf{p})$, $\mathbf{P}(\mathbf{q}, \mathbf{p})$ is a canonical transformation, $d\Phi$ is the total differential and the function $\Phi(\mathbf{q}, \mathbf{p})$ exists.

We will solve Eqs (1.1) for $\mathbf{x}$, $\mathbf{y}$, and Φ_y, Ψ_x, We obtain

$$\begin{gathered} \mathbf{x} = \frac{1}{2}(\mathbf{q} + \mathbf{Q}), \quad \mathbf{y} = \frac{1}{2}(\mathbf{p} + \mathbf{P}) \\ \Psi_{\mathbf{y}} = \mathbf{Q} - \mathbf{q}, \quad \Psi_{\mathbf{x}} = -\mathbf{P} + \mathbf{p} \end{gathered} \tag{2.3}$$

Hence, with the condition that the Jacobian of replacement (2.3) is non-zero ($\partial(\mathbf{x}, \mathbf{y})/\partial(\mathbf{q}, \mathbf{p}) \neq 0$), the equality $d\Phi = d\Psi$ follows and also the fact that the functions Φ and Ψ are identical:

$$\Psi(\mathbf{x}, \mathbf{y}) = \Psi\left(\frac{\mathbf{q} + \mathbf{Q}(\mathbf{q}, \mathbf{p})}{2}, \frac{\mathbf{p} + \mathbf{P}(\mathbf{q}, \mathbf{p})}{2}\right) = \Phi(\mathbf{q}, \mathbf{p})$$

It follows from relations (1.2) and (2.3) that

$$\frac{\partial(\mathbf{x}, \mathbf{y})}{\partial(\mathbf{q}, \mathbf{p})} = \frac{1}{J} = 2^{-2n}\det(E + A), \quad A = \frac{\partial(\mathbf{Q}, \mathbf{P})}{\partial(\mathbf{q}, \mathbf{p})}$$

and the condition that replacement (2.3) should be non-degenerate can be written as $\det(E + A) \neq 0$, were A is the Jacobi matrix and E is the identity matrix respectively.

We will formulate the result obtained.

Theorem 2. If in the region $(\mathbf{q}, \mathbf{p}) \in \Omega$ the transformation $\mathbf{Q}(\mathbf{q}, \mathbf{p})$, $\mathbf{P}(\mathbf{q}, \mathbf{p})$ is canonical, and not one of the eigenvalues of the Jacobi matrix A is equal to -1, parametrization (1.1) exists in the region Ω.

In [4] attention is drawn to the "depressing non-invariance" of generating functions with respect to the choice of the basis of the canonical system of coordinates and the invariance of the Poincaré differential form (2.2). Hence it follows that the parametric function $\Psi(\mathbf{x}, \mathbf{y})$ also has an invariant form. If the function $\Psi(\mathbf{x}, \mathbf{y})$ exists in some variables, it will also exist for any canonical replacement of the variables. The condition ($J \neq 0$) for the existence of parametric representation (1.1) is invariant with respect to the choice of the canonical variables, whereas the condition $\det S_{1\mathbf{qP}} \neq 0$ depends on the choice of the canonical variables. The condition $\det S_{1\mathbf{qP}} \neq 0$ may break down for a canonical replacement of the variables. Moreover, the class of parametrized canonical transformations is a considerably wider class of canonical transformations in terms of the generating function. Thus, rotation by 90°: $q = -p$ and $p = Q$ cannot be represented in terms of the generating function $S(q, P)$, but can be represented in terms of the parametric function $\Psi = x^2 + y^2$. These and other advantages of parametrization over the method of generating functions was pointed out in [3].

We will show how Eq. (1.3) leads to the previously developed [1, 2] method of invariant normalization of Hamiltonians.

3. INVARIANT NORMALIZATION OF HAMILTONIANS

In a Hamiltonian system the normal form of the Hamiltonian is called the normal Birkhoff form [5]. The most compact definition of this form can be found in [6]. In all cases the generating Hamiltonian is chosen in the form of the simplest quadratic form for a linear oscillatory system, and the definition of a normal form is tied to the choice of the generating Hamiltonian and has a non-invariant form [5–9].

Two methods of constructing canonical replacements, which reduce a system to normal form, are most widely used in the literature. One method is based on the use of generating functions. This was used by Birkhoff [5]. In other method, Lie generators are used as generating functions, which is more convenient, since it does not require the inversion of power series, that is necessary in the case of generating functions.

A general criterion of the Birkhoff normal form was proposed [1, 2] for a perturbed Hamiltonian

$$\bar{H}(t, \mathbf{q}, \mathbf{p}, \varepsilon) = H_0(t, \mathbf{q}, \mathbf{p}) + \bar{F}(t, \mathbf{q}, \mathbf{p}, \varepsilon), \quad \bar{F}(t, \mathbf{q}, \mathbf{p}, \varepsilon) = \varepsilon \bar{F}_1(t, \mathbf{q}, \mathbf{p}) + \varepsilon^2 \bar{F}_2(t, \mathbf{q}, \mathbf{p}) + \dots$$

Definition. A perturbed Hamiltonian has a normal form if and only if the perturbation is the first integral of the unperturbed part $\partial F/\partial t + \{H_0, F\} = 0$, where $\{f, g\} = f_{\mathbf{p}} g_{\mathbf{q}} - f_{\mathbf{q}} g_{\mathbf{p}}$ are Poisson brackets.

There are three reasons why this definition has an advantage over existing ones [4–8].

1. The solution of the complete system of Hamiltonian differential equations with the Hamiltonian in normal form is obtained by the superposition of the solutions of the unperturbed system and the solution of a system with an autonomous Hamiltonian, equal to $F(0, \mathbf{q}, \mathbf{p}, \varepsilon)$. The result was formulated in the form of a theorem in [2].

Zhuravlev's theorem. If a system with Hamiltonian $\bar{H}$ satisfies the condition for a normal form, then, to construct the general solution of the corresponding Hamiltonian equations, it is sufficient:

(A) to obtain a general solution of the generating system with Hamiltonian $H_0(t, p, q)$;

(B) to obtain a general solution of the system, defined solely by the perturbation $F(0, p, q, \varepsilon)$, with the condition that, in this system, the time, which occurs explicitly in the Hamiltonian, is put equal to zero.

The general solution of the initial non-autonomous system can then be represented by the composition of the solution obtained in arbitrary order (instead of arbitrary constants in the solution of the second system one substitutes the solution of the first or vice versa).

2. The invariant form of the criterion enables one to carry out normalization without preliminary simplification of the unperturbed part and without separation in the case of autonomous–non-autonomous and resonance–non-resonance.

3. The asymptotics of the normal form and the replacement of the variables, which reduce the Hamiltonian to normal form, are found by successive quadratures of the functions known at each step.

4. THE ALGORITHM OF INVARIANT NORMALIZATION USING PARAMETRIC REPLACEMENT

We will show how Eq. (1.3) of Theorem 1 can be converted to an analogue of Zhuravlev's normalization method.

Suppose we are given the Hamiltonian

$$H(t, \mathbf{q}, \mathbf{p}) = H_0(t, \mathbf{q}, \mathbf{p}) + F(t, \mathbf{q}, \mathbf{p}, \varepsilon), \quad F(t, \mathbf{q}, \mathbf{p}, \varepsilon) = \varepsilon F_1(t, \mathbf{q}, \mathbf{p}) + \varepsilon^2 F_2(t, \mathbf{q}, \mathbf{p}) + \dots$$

which it is required to convert to normal form. Suppose $\bar{H}^{(k)}(t, \mathbf{Q}, \mathbf{P}, \varepsilon) = H_0(t, \mathbf{Q}, \mathbf{P}) + \bar{F}^{(k)}(t, \mathbf{Q}, \mathbf{P}, \varepsilon)$ is the asymptotic form of the kth order normal form $\bar{F}^{(k)}(t, \mathbf{q}, \mathbf{p}, \varepsilon) = \varepsilon \bar{F}_1(t, \mathbf{Q}, \mathbf{P}) + \dots + \varepsilon^k \bar{F}_k(t, \mathbf{Q}, \mathbf{P})$ with the canonical replacement (1.1) and $\Psi^{(k)}(t, \mathbf{x}, \mathbf{y}, \varepsilon) = \varepsilon \Psi_1(t, \mathbf{x}, \mathbf{y}) + \dots + \varepsilon^k \Psi_k(t, \mathbf{x}, \mathbf{y})$ is the asymptotic form of the kth of the function $\Psi(t, \mathbf{x}, \mathbf{y}, \varepsilon)$ in relations (1.1).

Then, by Theorem 1 the asymptotic form $\Psi^{(k)}$ will satisfy Eq. (1.3) which can be written as

$$\begin{aligned} &\frac{\partial \Psi^{(k)}}{\partial t} + H_0\left(t, \mathbf{x} - \frac{1}{2}\Psi^{(k)}_{\mathbf{y}}, \mathbf{y} + \frac{1}{2}\Psi^{(k)}_{\mathbf{x}}\right) - H_0\left(t, \mathbf{x} + \frac{1}{2}\Psi^{(k)}_{\mathbf{y}}, \mathbf{y} - \frac{1}{2}\Psi^{(k)}_{\mathbf{x}}\right) + \\ &+ F^{(k)}\left(t, \mathbf{x} - \frac{1}{2}\Psi^{(k)}_{\mathbf{y}}, \mathbf{y} + \frac{1}{2}\Psi^{(k)}_{\mathbf{x}}\right) = \bar{F}^{(k)}\left(t, \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}}, \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}\right) \end{aligned} \tag{4.1}$$

Hence follows the chain of equations for determining the coefficients of the expansions of the canonical replacements Ψ_i and of the normalized Hamiltonians $\bar{F}_i$

$$\frac{\partial \Psi_i}{\partial t} + \{H_0, \Psi_i\} + R_i = \bar{F}_i, \quad \frac{\partial \bar{F}_i}{\partial t} + \{H_0, \bar{F}_i\} = 0; \quad i = 1, 2, \dots \tag{4.2}$$

The functions R_i are calculated successively from the formulae

$$R_1 = F_1, \quad R_2 = F_2 + \frac{1}{2}\{F_1 + \bar{F}_1, \Psi_1\}, \dots \tag{4.3}$$

If H_0 is a polynomial of no higher than the second degree in $\mathbf{q}$ and $\mathbf{p}$, then R_i, $i \leqslant k$ will be coefficients of the expansion in powers of ε of the function

$$F\left(t, \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}}, \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}\right) - \bar{F}^{(k)}\left(t, \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}}, \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}\right) + \bar{F}^{(k)}(t, \mathbf{x}, \mathbf{y}) = \\ = \varepsilon R_1 + \varepsilon^2 R_2 + \varepsilon^3 R_3 + \dots \tag{4.4}$$

A similar chain of equations (4.2) was obtained previously in [1, 2]; the equations are called homological equations and are written in the following form

$$R_i = \bar{F}_i - \frac{d\Psi_i}{dt}, \quad \frac{d\bar{F}_i}{dt} = 0; \quad i = 1, 2, \dots \tag{4.5}$$

Here the total derivatives d/dt are calculated using the rule of the differentiation of complex functions $\Psi_i(t, \mathbf{x}, \mathbf{y})$, $F_i(t, \mathbf{x}, \mathbf{y})$, in which $\mathbf{x}(t)$ and $\mathbf{y}(t)$, as functions of time, are determined from the solution of the problem for the unperturbed system

$$\dot{\mathbf{x}} = H_{0_\mathbf{y}}, \quad \dot{\mathbf{y}} = -H_{0_\mathbf{x}}, \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{4.6}$$

If, in relation (4.5), we substitute the solution of system (4.6) instead of $\mathbf{x}$ and $\mathbf{y}$, then it follows from the second equation of (4.5) that the function $\bar{F}_i$ is independent of the time t. The integral with respect to time of the first equation will then have the form

$$\int_{t_0}^{t} R_i(t)dt = (t - t_0)\bar{F}_i(t_0, \mathbf{x}_0, \mathbf{y}_0) + \Psi_i(t_0, \mathbf{x}_0, \mathbf{y}_0) - \Psi_i(t, \mathbf{x}, \mathbf{y}) \tag{4.7}$$

It also gives a key to the complete solution of the problem: the quadrature (4.7) also defines the normal form of the function Ψ_i in the replacement of variables (1.1).

Unfortunately, it is not always possible to represent the integral of the function R_i in the form (4.7) uniquely. There will be uniqueness if the function R_i, after substituting solution of system (4.6) into it, turns out to be quasi-periodic (the sum of functions periodic in t). In this case the integral of R_i is equal to a linear and quasi-periodic function $f(t)$. One can subtract from $f(t)$ the mean part $\bar{f}(t)$, which is independent of time, and relate it to the second term on the right-hand side of (4.7). Representation (4.7) will then uniquely define $\bar{F}_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$ and the function $\Psi_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$ with zero mean-time value $\overline{\Psi_i(t, \mathbf{x}(t), \mathbf{y}(t))} = 0$. The condition for R_i to be quasi-periodic imposes limitations on the parameters for which a normal form exists.

We will formulate the result obtained.

The fundamental theorem. Asymptotic forms of the kth approximation of the normal form, and of the replacement of variables which lead to it, exist and are unique, if, after substituting the solution of system (4.6) into the functions $R_i (i = 1, 2, \dots, k)$, they turn out to be functions that are quasi-periodic with time. Then, on the right-hand side of integral (4.7) $\bar{F}_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$ is a coefficient of the term that is linear in t and $\bar{\Psi}_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$ is a term that is independent of time.

We will list the mean features of the proposed algorithm, which distinguish it from the main features of Zhuravlev's algorithm [1, 2], given in braces.

1. The initial system $H(t, \mathbf{q}, \mathbf{p})$ is non-autonomous. {System $H(\mathbf{q}, \mathbf{p})$ is autonomous; if the initial system is non-autonomous, then, at first, it is reduced to an autonomous system with an increase in its order.}

2. The function $\Psi(t, \varepsilon, \mathbf{x}, \mathbf{y})$ is used for the canonical replacement. {For this purpose the generating Hamiltonian $G(\varepsilon, \mathbf{Q}, \mathbf{P})$ is used.}

3. The canonical replacement $(\mathbf{q}, \mathbf{p}) \Rightarrow (\mathbf{Q}, \mathbf{P})$ is sought in the parametric form (1.1). {The canonical replacement is sought on the phase flux of the Hamiltonian system.}

The relation between the generating Hamiltonian G and the function Ψ. In Zhuravlev's method [1, 2] the replacement $\mathbf{q}, \mathbf{p} \to Q, \mathbf{P}$ is sought on the phase flux of the Hamiltonian system

$$\begin{aligned} &d\mathbf{X}/d\tau = G_Y, \quad dY/d\tau = -G_X \\ &\mathbf{X}(0) = \mathbf{q}, \quad \mathbf{Y}(0) = \mathbf{p}; \quad \mathbf{X}(\varepsilon) = \mathbf{Q}, \quad \mathbf{Y}(\varepsilon) = \mathbf{P} \end{aligned} \tag{4.8}$$

where $\tau \in [0, \varepsilon]$ is an auxiliary parameter similar to the time t.

This replacement in the proposed algorithm is carried out using parameterization. The function Ψ, which defines the mapping onto the phase flux of the Hamiltonian system (4.8), if found from the solution of the problem

$$\Psi_\tau(\tau, \mathbf{x}, \mathbf{y}) = G\left(\mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}}, \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}\right), \quad \Psi(0, \mathbf{x}, \mathbf{y}) = 0$$

We obtain $\Psi = \varepsilon G$, to an accuracy of $\tau^3 = \varepsilon^3$. Hence, the asymptotic forms $\Psi_1 = G_0$, $\Psi_2 = G_1$ of the first two approximations in both methods are identical, and consequently R_1 and R_2 in both methods are also identical. The remaining approximations for $R_3, R_4, \ldots$ will be different. The normal form itself is independent of the choice of the method.

5. THE ALGORITHM OF INVARIANT NORMALIZATION FOR THE ASYMPTOTIC DETERMINATION OF THE SEQUENCE OF POINCARÉ POINTS

The algorithm proposed above enables one to obtain the asymptotic form of the general solution of the kth order. The algorithm can be considerably simplified for Hamiltonians that are periodic in time. In this case, rather than obtain the trajectories of motion $\mathbf{q}(t)$ and $\mathbf{p}(t)$, it is more useful to separate out in them the sequence of points $\mathbf{q}_m = \mathbf{q}(Tm)$, $\mathbf{p}_m = \mathbf{P}(Tm)$ corresponding to instants of time $t = t_m = Tm$ $(m = 0, 1, 2, \ldots)$ that are multiples of the period T. We will call this sequence of points on the trajectory Poincaré points.

The asymptotic solution for Poincaré points is constructed as follows. From quadrature (4.7) when $i = 1, \ldots, k$ we obtain the functions $\bar{F}_i(0, \mathbf{Q}, \mathbf{P})$ and $\Psi_i(0, \mathbf{x}, \mathbf{y})$, where the last quadrature can be simplified by assuming $t_0 = 0$ in equality (4.7). Hence, we obtain the asymptotic forms of the kth approximation

$$\bar{F}^{(k)}(0, \mathbf{Q}, \mathbf{P}, \varepsilon) = \varepsilon \bar{F}_1(0, \mathbf{Q}, \mathbf{P}) + \ldots + \varepsilon^k F_k(0, \mathbf{Q}, \mathbf{P})$$

$$\bar{\Psi}^{(k)}(0, \mathbf{x}, \mathbf{y}, \varepsilon) = \varepsilon \bar{\Psi}_1(0, \mathbf{x}, \mathbf{y}) + \ldots + \varepsilon^k \Psi_k(0, \mathbf{x}, \mathbf{y})$$

After this we use Zhuravlev's theorem.

Suppose $\mathbf{Q}(Tm, \mathbf{a}, \mathbf{b})$, $\mathbf{P}(Tm, \mathbf{a}, \mathbf{b})$ are the Poincaré points of the unperturbed system. Suppose $\mathbf{X}(Tm, \mathbf{Q}_0, \mathbf{P}_0)$, $\mathbf{Y}(Tm, \mathbf{Q}_0, \mathbf{P}_0)$ are the Poincaré points, obtained from the solution of the system of equations

$$\dot{\mathbf{X}} = \frac{\partial}{\partial \mathbf{Y}} \bar{F}^{(k)}(0, \mathbf{X}, \mathbf{Y}, \varepsilon), \quad \dot{\mathbf{Y}} = -\frac{\partial}{\partial \mathbf{X}} \bar{F}^{(k)}(0, \mathbf{X}, \mathbf{Y}, \varepsilon), \quad \mathbf{X}(0) = \mathbf{Q}_0, \quad \mathbf{Y}(0) = \mathbf{P}_0$$

Then the Poincaré $\mathbf{Q}_m$ and $\mathbf{P}_m$ of the complete Hamiltonian system in the new variables $\mathbf{Q}$ and $\mathbf{P}$ are obtained by substituting $\mathbf{a} = \mathbf{X}(Tm, \mathbf{Q}_0, \mathbf{P}_0)$, $\mathbf{b} = \mathbf{Y}(Tm, \mathbf{Q}_0, \mathbf{P}_0)$ into the functions $\mathbf{Q}(Tm, \mathbf{a}, \mathbf{b})$, $\mathbf{P}(Tm, \mathbf{a}, \mathbf{b})$ $(m = 0, 1, \ldots)$.

In the original variables the Poincaré points are obtained using parametric replacement with the function $\bar{\Psi}^{(k)}(0, \mathbf{x}, \mathbf{y}, \varepsilon)$. In this replacement, the parameters $\mathbf{x}$ and $\mathbf{y}$ can be eliminated by expressing them in terms of $\mathbf{q}$ and $\mathbf{p}$

$$\mathbf{x}(\mathbf{q}, \mathbf{p}) = \mathbf{q} + \frac{1}{2}\Psi_{\mathbf{p}}(\mathbf{q}, \mathbf{p}) + \frac{1}{4}\{\Psi, \Psi_{\mathbf{p}}\} + \ldots$$

$$\mathbf{y}(\mathbf{q}, \mathbf{p}) = \mathbf{p} - \frac{1}{2}\Psi_{\mathbf{q}}(\mathbf{q}, \mathbf{p}) - \frac{1}{4}\{\Psi, \Psi_{\mathbf{q}}\} + \ldots$$

The new variables are expressed in terms of the old ones as follows:

$$\mathbf{Q} = 2\mathbf{x}(\mathbf{q}, \mathbf{p}) - \mathbf{q}, \ \mathbf{P} = 2\mathbf{y}(\mathbf{q}, \mathbf{p}) - \mathbf{p}$$

As a result, we obtain the relation between the new variables and the old ones

$$\begin{aligned} \mathbf{Q}(\mathbf{q}, \mathbf{p}) &= \mathbf{q} + \Psi_{\mathbf{p}}(\mathbf{q}, \mathbf{p}) + \frac{1}{2}\{\Psi, \Psi_{\mathbf{p}}\} + \dots \\ \mathbf{P}(\mathbf{q}, \mathbf{p}) &= \mathbf{p} - \Psi_{\mathbf{q}}(\mathbf{q}, \mathbf{p}) - \frac{1}{2}\{\Psi, \Psi_{\mathbf{q}}\} + \dots \end{aligned} \tag{5.1}$$

In order to express the old variables in terms of the new ones, it is sufficient in formulae (5.1) to replace them with one another and then change the sign of Ψ into the opposite. We obtain

$$\begin{aligned} \mathbf{q}(\mathbf{Q}, \mathbf{P}) &= \mathbf{Q} - \Psi_{\mathbf{P}}(\mathbf{Q}, \mathbf{P}) + \frac{1}{2}\{\Psi, \Psi_{\mathbf{P}}\} + \dots, \\ \mathbf{p}(\mathbf{Q}, \mathbf{P}) &= \mathbf{P} + \Psi_{\mathbf{Q}}(\mathbf{Q}, \mathbf{P}) - \frac{1}{2}\{\Psi, \Psi_{\mathbf{Q}}\} + \dots \end{aligned} \tag{5.2}$$

Note that in the invariant normalization method [1, 2] the Campbell–Hausdorff formulae are used for this purpose, which, to a second approximation, are identical with formulae (5.1) and (5.2), apart from the replacement of Ψ by the generating Hamiltonian G.

6. EXAMPLES OF ASYMPTOTIC SOLUTIONS

Extremely instructive examples [1, 2] demonstrate the considerable simplifications compared to all previous ones. This method is equivalent in simplicity to Zhuravlev's method [1, 2], but differs from it in that the chain of equations for the asymptotic forms is written in the initial Hamiltonian system irrespective of whether the system is autonomous or non-autonomous. In Zhuravlev's method [1, 2] the non-autonomous system must be reduced to an autonomous system with an increase in the order of the system and a chain of equations for the asymptotic forms is then written for it.

We will demonstrate the method using two examples of the solution of problems of forced oscillations in resonance. To solve these problems by the classical method it is necessary to introduce another definition of a normal form [6]. This is not required in the proposed method. The normal form is calculated directly from the quadrature and the solution is then found. We will show this.

Example 1. It is required to obtain a general solution of the equation describing the forced oscillations of a linear oscillator at resonance: $\ddot{q} + q = \varepsilon \sin t$.

Example 2. In the problem of the forced oscillations of a non-linear Duffing oscillator $\ddot{q} + q = \varepsilon(\sin t - q^3 + 2\lambda q)$, it is required to obtain the value of λ for which the solution is periodic in time with period 2π and to investigate the stability of this solution.

In both examples the equations are Hamiltonian, have the same unperturbed Hamiltonian $H_0 = (q^2 + p^2)/2$ and the same solution of the unperturbed system of equations corresponding to it

$$q = q_0 \cos(t - t_0) + p_0 \sin(t - t_0), \quad p = -q_0 \sin(t - t_0) + p_0 \cos(t - t_0) \tag{6.1}$$

It is basic for constructing the normal forms in both examples.

We will obtain the first approximation in Example 1. Substituting solution (6.1) into the expressions $R_1 = F_1 = -q \sin t$, we obtain a function $R_1(t)$, periodic in time, the integral of which has the form

$$\int_{t_0}^{t} R_1(t)dt = \bar{F}_1(t_0, q_0, p_0)(t - t_0) + \bar{\Psi}_1(t_0, q_0, p_0) + f(t) =$$

$$= -\frac{1}{2}(q_0 \sin t_0 + p_0 \cos t_0)(t - t_0) - \frac{1}{4}(q_0 \cos t_0 + p_0 \sin t_0) + f(t)$$

Hence we obtain the first coefficients $\bar{F}_1$ and $\bar{\Psi}_1$ and expansions of the normal form $\bar{H} = H_0 + \varepsilon \bar{F}_1$ and of the function $\bar{\Psi} = \varepsilon \Psi_1$

$$\bar{H} = \frac{1}{2}(Q^2 + P^2) + \bar{F}(t, Q, P, \varepsilon), \quad \bar{F}(t, Q, P, \varepsilon) = -\frac{\varepsilon}{2}(Q\sin t + P\cos t)$$

$$\Psi(t, Q, P, \varepsilon) = -\frac{\varepsilon}{4}(Q\cos t + P\sin t)$$

We will obtain the solution of the first approximation using Zhuravlev's theorem [2]. The system $\dot{Q} = -\varepsilon/2$, $\dot{P} = 0$ corresponds to the perturbed part of the Hamiltonian $\bar{F}(0, Q, P) = -\varepsilon P/2$, and the solution

$$Q = Q_0 - \varepsilon t/2, \quad P = P_0$$

We substitute Q and P instead q_0 and p_0 into solution (6.1) and put $t_0 = 0$ in it. We obtain a solution of the system of equations in the variables Q and P

$$Q = (Q_0 - \varepsilon t/2)\sin t + P_0 \sin t, \quad P = -(Q_0 - \varepsilon t/2)\sin t + P_0 \cos t$$

The solution of the system of equations in the original variables q and p is obtained as follows. Substituting the function $\Psi(t, Q, P, \varepsilon)$ into formula (5.2), we obtain the replacement

$$q = Q + \frac{\varepsilon}{4}\sin t, \quad p = P - \frac{\varepsilon}{4}\cos t$$

and, using it, we obtain the solution in the original variables

$$q = \left(Q_0 - \frac{\varepsilon}{2}t\right)\cos t + \left(P_0 + \frac{\varepsilon}{4}\right)\sin t = \left(q_0 - \frac{\varepsilon}{2}t\right)\cos t + \left(p_0 + \frac{\varepsilon}{2}\right)\sin t$$

In the linear problem all the next terms of the series in ε are equal to zero, and hence the first approximation is the exact solution.

The solution of Example 2 was obtained previously in [1, 2] by the averaging method. For comparison we will obtain the solution by the normal-form method.

We again use the quadrature (4.7). Substituting solution (6.1) into the expression $R_1 = F_1 = -q\sin t - \lambda p^2 + q^4/4$, we obtain the integrand $R_1(t)$. From quadrature (4.7) we obtain the coefficient of the expansion of the normal form $\bar{F}_1(t, Q, P)$ and, putting $t = 0$ in it, we obtain

$$\bar{F}_1(0, Q, P) = -\frac{1}{2}P - \frac{\lambda}{2}(Q^2 + P^2) + \frac{3}{32}(Q^2 + P^2)^2$$

A fixed point will correspond to the periodic solution. Its coordinates are Q and P, which satisfy the system

$$\frac{\partial \bar{F}_1}{\partial Q} = Q\left(-\lambda + \frac{3}{8}A^2\right) = 0, \quad \frac{\partial \bar{F}_1}{\partial P} = -\frac{1}{2} + P\left(-\lambda + \frac{3}{8}A^2\right) = 0$$

Hence we obtain $Q = 0$, $P = \pm A$ when $\lambda = \dfrac{3}{8}A^2 \pm \dfrac{1}{2A}$, where $A = \sqrt{Q^2 + P^2}$ is the amplitude. The dependence of $\omega = 1 - \varepsilon\lambda$ on A is called the amplitude–frequency characteristic.

The fixed point will be stable if the function $\bar{F}_1$ in it reaches a strict minimum or maximum. Hence we obtain the condition for the periodic solution to be stable

$$\left(\lambda - \frac{3}{8}A^2\right)\left(\lambda - \frac{9}{8}A^2\right) > 0$$

which agrees with the similar condition obtained by the averaging method.

In the third example, given below, to obtain a solution it is necessary to find higher approximations. The solution by the classical method would involve lengthy calculations. Using our method a solution is obtained much more simply.

Example 3. It is required to obtain the Poincaré points corresponding to the instants of time $t_n = 2\pi n$ for the non-linear equation $\ddot{q} = \varepsilon^2 \cos t' \cos q$ up to terms $O(\varepsilon^6)$.

This equation describes different problems in mechanics and physics. One of them is the vibrational motion of a spherical particle in a liquid, in which a plane standing acoustic wave is produced [9, 10]. Suppose we have a vertical tube with a rigid horizontal cover. A standing wave is excited in the tube, in which the velocity of the liquid varies as $v = A\omega\sin\omega t\cos kz$, where ω is the frequency of the wave, k is the wave number, z is the axis, directly vertically upwards and A is the amplitude of displacement of the particles of the liquid, which is assumed to be small. The frequency and wave number are related to the velocity of sound in the liquid as $\omega = kc$. If the condition $\mu/(\rho kca^2) \ll 1$ is satisfied for a particle of radius a, the Stokes friction force and the Basset hereditary force will be negligibly small compared with the inertia forces. The equation of motion of a particle then has the form

$$(\rho + 2\rho_0)\ddot{z}_0 = 3\rho w - 2(\rho_0 - \rho)g; \quad w = \partial v + v\partial v/\partial z \approx \partial v/\partial t = A\omega^2\cos\omega t\cos kz$$

Here ρ and ρ_0 are the densities of the liquid and the solid particle and μ is the coefficient of dynamic viscosity of the liquid.

For a particle of neutral buoyancy ($\rho = \rho_0$) the equation reduces to the equation of Example 3, in which $q = kz_0$, $t' = \omega t$, $\varepsilon = Ak$.

The classical averaging method [11] has been used to solve this equation [9, 10]. Expansion was carried out with respect to the parameter ε. To investigate the problem three approximations are required. The solution is obtained in the form

$$q = \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + O(\varepsilon^4).$$

We will show how the solution can be obtained using our method. The expansion will be carried out with respect to the parameter $\delta = \varepsilon^2$. Therefore, to achieve much higher accuracy, of the order of ε^6, two approximations in all are required with much fewer calculations.

The system of Hamiltonian equations with Hamiltonian function

$$H = \frac{1}{2}p^2 + \delta F_1(t, q, p); \quad F_1 = -\cos t\sin q$$

is reduced to the equation of the example.

We obtain the solution of the unperturbed system

$$q = q_0 + p_0(t - t_0), \quad p = p_0$$

and we substitute it into the expression $R_1 = F_1$. We substitute the periodic function $R_1(t)$ obtained into the quadrature (4.7). We have

$$\int_{t_0}^{t} R_1 dt = -\frac{\cos(t_0 + q_0)}{2 + 2p_0} + \frac{\cos(-t_0 + q_0)}{2 - 2p_0} + f_1(t)$$

Hence we obtain the first approximation of the normal form and the parametric replacement of variables

$$\bar{F}_1 = 0, \quad \bar{\Psi}_1(t, q, p) = -\frac{\cos(t + q)}{2 + 2p} + \frac{\cos(-t + q)}{2 - 2p} \tag{6.2}$$

In the second approximation

$$R_2 = -\frac{1}{2}\frac{\partial F_1}{\partial q}\frac{\partial \bar{\Psi}_1}{\partial p} = \frac{1}{4}\cos t\cos q\left(\frac{\cos(t + q)}{(1 + p)^2} + \frac{\cos(t - q)}{(1 - p)^2}\right)$$

From integral (4.7) we obtain a term, equal to F_2, which is linear with respect to time, and a term, equal to Ψ_2, which is independent of the time. The final form of the normal form and of the function which defines the parametric replacement are

$$\bar{H} = \frac{1}{2}P^2 + \frac{\delta^2}{16}\left[\frac{1}{(1 + P)^2} + \frac{1}{(1 - P)^2}\right] + O(\delta^3)$$

$$\Psi(0, x, y) = \frac{\delta y}{1 - y^2}\cos x - \frac{\delta^2(1 - 3y^2 - 2y^4)}{16y(1 - y^2)^3}\sin 2x + O(\delta^3)$$

The function $\Psi(0,x,y)$ has a denominator which equal to zero for $y = 0, \pm 1$. Because of this, the normal form obtained is unsuitable for analysing motion with a small momentum $p \sim \delta$. This well-known problem of small denominators of the asymptotic theory for Hamiltonian systems arises due to the fact that the unperturbed Hamiltonian is unsuitably chosen and does not reflect the qualitative behaviour of the system for a small momentum. To eliminate the small denominator in the formulae of the replacement of variables of the second approximation, one must change the unperturbed Hamiltonian and its change is taken to be a perturbation.

We will illustrate this general rule using the solution of the example considered.

Elimination of the small denominator $y = 0$. In order to eliminate the small denominator in the function $\Psi_2(x,y)$ of Example 3, we will represent the unperturbed Hamiltonian H_0 and the perturbation F in the form

$$H_0 = \frac{1}{2}p^2 + \frac{\delta^2}{8}\cos 2q, \quad F = -\delta\cos t\sin q - \frac{\delta^2}{8}\cos 2q$$

Obviously the Hamiltonian of the system $H = H_0 + F$ remains unchange.

We will obtain, by the invariant normal from method, the asymptotic solution for the Poincaré points apart from small terms of the order of $\delta^3 = \varepsilon^6$. We will represent the solution of the unperturbed system in the form of an expansion in the parameter δ

$$q = a + bt + \delta^2\left(\frac{t}{4b}\cos 2a - \frac{1}{4b^2}(\sin(a+bt) - \sin a)\right) + O(\delta^3)$$

$$p = b + \frac{\delta^2}{4b}(\cos a - \cos(a+bt)) + O(\delta^3); \quad a = q(0), \quad b = p(0), \quad b \neq 0$$

From the quadrature of the first approximation we obtain the same as in the previous solution (6.2). In expression for R_2 we added the term

$$F_2 = -\frac{1}{8}\cos 2q = -\frac{1}{8}\cos(2a + 2bt)$$

The quadrature of R_2 is correspondingly changed. Calculating it, we obtain the normal form

$$\bar{H} = \frac{1}{2}P^2 + \delta^2\left(P^2\frac{3-P^2}{8(1-P^2)^2} - \frac{1}{4}\sin^2 Q\right) + O(\delta^3) \tag{6.3}$$

and the function, which defines the parametric replacement, with the eliminated singularity as $y \to 0$

$$\Psi(0,x,y) = \delta\Psi_1(0,x,y) + \delta^2\Psi_2(0,x,y) + O(\delta^3)$$

$$\left(\Psi_1(0,x,y) = \frac{y}{(1-y^2)}\cos x, \quad \Psi_2(0,x,y) = \frac{y^3(5-y^2)}{16(1-y^2)^3}\sin 2x\right) \tag{6.4}$$

The normal form enables us to obtain the fixed points and to investigate their stability. From the equations

$$\partial\bar{H}/\partial Q = 0, \quad \partial\bar{H}/\partial P = 0 \Rightarrow P = 0, \quad \sin Q\cos Q = 0$$

we determine, in the period $Q \in [0, 2\pi)$, the coordinates of four fixed points $M_i(Q_i, P_i) = ((i-1)\pi/2, 0)$ $(i = 1, 2, 3, 4)$. The points M_2 and M_4 correspond to a minimum of the function $\bar{H}$ and, by Lagrange's theorem, are stable. By Lyapunov's theorem they correspond to a stable periodic solution. M_1 and M_3 are hyperbolic-type points. The function $\bar{H}$ has no minimum or maximum in them, and consequently these points are unstable. They correspond to two unstable periodic solutions.

The invariant curves on which the Poincaré points lie are obtained from the integral of the normal form $\bar{H}(Q_m, P_m) = E$. The invariant curves in the neighbourhood of the fixed points, in which $E \sim \delta^2$, are of greatest interest. We will make the replacement $E = \delta^2 C/4$. Then, apart from small terms of the order of δ^3 we obtain a single-parameter family of invariant curves for O_m and P_m $(m = 0, 1, 2, \ldots)$

$$P_m^2\left(1+\delta^2\frac{3-P_m^2}{4(1-P_m^2)^2}\right) = \frac{1}{2}\delta^2(C+\sin^2 Q_m)^2 \Rightarrow P_m = \pm\delta\left(1-\frac{3}{8}\delta^2\right)\sqrt{\frac{1}{2}(C+\sin^2 Q_m)}$$

Using the replacement of variables

$$p = P\left(1-\delta\sin Q+\frac{1}{2}\delta^2+O(\delta^3)\right)$$

which follows from (4.10) and (4.15), we obtain the invariant curves in the initial variables

$$p_m = \delta\left(1-\delta\sin Q_m+\frac{1}{8}\delta^2\right)\sqrt{\frac{1}{2}(C+\sin^2 Q_m)}$$

The new variable Q must be expressed in terms of the old one using the replacement

$$Q_m = q_m+\delta\cos q_m-\frac{1}{4}\delta^2\sin 2q_m+O(\delta^3)$$

which is found from the formulae of the replacement of variables (5.1). For $\delta < 0.1$ it is sufficient to use the less accurate approximation

$$p_m = \pm\delta(1-\delta\sin q_m)\left[\frac{1}{2}(C+\sin^2(q_m+\delta\cos q_m))\right]^{1/2} \tag{6.5}$$

In the range $-1 \leqslant C < 0$ the Poincaré points lie on closed invariant curves. They correspond to finite motion around the fixed point M_2 or M_4. If we take into account an infinitesimal friction force, these invariant curves become spirals, along which the Poincaré points will tend either to M_2 or M_4 as $t \to \infty$.

When $C > 0$ the invariant curves correspond to infinite motion. When $C = 0$ we obtain the equation of the separatrices

$$p_m = \pm\frac{\delta}{\sqrt{2}}(1-\delta\sin q_m)|\sin(q_m+\delta\cos q_m)|$$

which separate the finite and infinite motions.

In Fig. 1 we show the Poincaré points, obtained numerically from the original equations with $\delta = 0.16$. The initial values of the Poincaré points are denoted by the small circles. These lie on invariant curves which are indistinguishable from the curves defined by Eq. (6.5). The dark circles denote the positions of the fixed points $M_i (i = 1, 2, 3, 4)$. Their coordinates in the variables q, p are $M_1(2\pi-\delta, 0)$, $M_2(\pi/2, 0)$, $M_3(\pi+\delta, 0)$, $M_4(3\pi/2, 0)$.

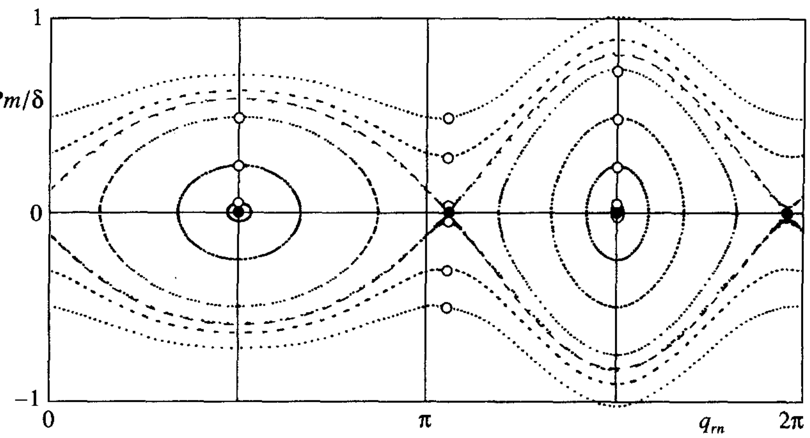

Fig. 1

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